Strategy and Politics:
Mixed Strategy Nash Equilibrium

Matt Golder

Pennsylvania State University

Notes

Nash Equilibrium

A Nash equilibrium of a strategic game is an action profile in which every player's action is optimal given every other player's action.

Such a profile represents a steady state: every player's behavior is the same whenever she plays the game, and no player wishes to change her behavior.

Notes

Nash Equilibrium

A Nash equilibrium of a strategic game is an action profile in which every player's action is optimal given every other player's action.

Such a profile represents a steady state: every player's behavior is the same whenever she plays the game, and no player wishes to change her behavior.

More general notions of steady state allow the players' choices to vary, as long as the **pattern** of choices remains constant.

For example, each individual may, on each occasion she plays the game, choose her action probabilistically according to the same, unchanging distribution.

- In each play of the game the individual plays an action *a* with the same probability p.

This leads us to the notion of a **stochastic steady state.**

Notes

Nash Equilibrium

This notion of a *stochastic* steady state can be modeled by a **mixed strategy Nash equilibrium**, a generalization of the notion of Nash equilibrium.

- So far, we have examined **pure strategy** Nash equilibrium in which the equilibrium actions are chosen with probability 1.
- Strictly speaking, a pure strategy is just a special case of a mixed strategy, where the action is chosen with probability $p = 1$ (from all possible $0 \geq p \geq 1$).

Notes

von Neumann-Morgenstern Preferences

Once players begin randomizing – choosing actions probabilistically – we can't use ordinal preferences any longer.

If players don't randomize, preferences can be defined over action profiles.

But if they do randomize, they need preferences over *lotteries* over action profiles.

Preferences over lotteries over action profiles can be expressed as expected utilities over the outcomes and are called von Neumann-Morgenstern (vNM) preferences.

Notes

Strategic Games with vNM Preferences

A **strategic form game** with vNM preferences consists of

1. a set of players, N
2. for each player i, a set of actions, A_i
3. for each player i, vNM preferences

In a stochastic steady state of a strategic game, we allow each player to choose a probability distribution over her set of actions rather than restricting her to choose a deterministic action.

A **mixed strategy** of a player in a strategic game is a probability distribution over the player's actions.

Notes

Some Notation

We need some notation:

- α is a mixed-strategy profile.
- $\alpha_i(a_i)$ is a probability assigned to action a_i by α_i.
- α_i is a mixed strategy for player i.

To specify a mixed strategy of player i we need to give the probability it assigns to each of player i's actions.

- For example, the strategy of player 1 in a Matching Pennies game that assigns probability $\frac{1}{2}$ to each action is written as the strategy α_1 for $\alpha_1(Head) = \frac{1}{2}$ and $\alpha_1(Tail) = \frac{1}{2}$.
- A shorthand for this that is often used: player 1's mixed strategy is $(\frac{1}{2}, \frac{1}{2})$.

Notes

Some Notation

Figure: Matching Pennies

		Player 2	
		Heads	Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

A mixed strategy profile in the Matching Pennies game would be

$$\alpha = (\alpha_1; \alpha_2)$$
$$\alpha = [(\alpha_1(Heads), \alpha_1(Tails)); ((\alpha_2(Heads), \alpha_2(Tails))]$$

Notes

Mixed Strategy Nash Equilibrium

A mixed strategy may assign probability 1 to a single action:

- By allowing a player to choose probability distributions, we do not prohibit her from choosing deterministic actions.
- We refer to such a mixed strategy as a **pure strategy**.
- Player i choosing the pure strategy that assigns probability 1 to the action a_i is equivalent to her simply choosing the action a_i.

We can now define a mixed strategy Nash equilibrium.

Notes

Mixed Strategy Nash Equilibrium

Mixed Strategy Nash Equilibrium

The mixed strategy profile α^* in a strategic game with vNM preferences is a mixed strategy Nash equilibrium if, for each player i and every mixed strategy α_i of player i, the expected payoff to player i of α^* is at least as large as the expected payoff to player i of $(\alpha_i, \alpha^*_{-i})$ according to a payoff function whose expected value represents player i's preferences over lotteries.

Equivalently, for each player i,

$$E[u_i(\alpha^*)] \geq E[u_i(\alpha_i, \alpha^*_{-i})] \quad \text{for every mixed strategy } \alpha_i \text{ of player } i$$

Notes

Mixed Strategy Nash Equilibrium

Mixed Strategy Nash Equilibrium

The mixed strategy profile α^* in a strategic game with vNM preferences is a mixed strategy Nash equilibrium if, for each player i and every mixed strategy α_i of player i, the expected payoff to player i of α^* is at least as large as the expected payoff to player i of $(\alpha_i, \alpha^*_{-i})$ according to a payoff function whose expected value represents player i's preferences over lotteries.

Equivalently, for each player i,

$$E[u_i(\alpha^*)] \geq E[u_i(\alpha_i, \alpha^*_{-i})] \quad \text{for every mixed strategy } \alpha_i \text{ of player } i$$

One way to find a mixed strategy Nash equilibrium is to use **best response functions**.

Notes

Best Response Functions

Denote player i's best response function by B_i.

For a strategic game with ordinal preferences, $B_i(a_{-i})$ is the set of player i's best actions when the list of the other players' actions is a_{-i}.

For a strategic game with vNM preferences, $B_i(\alpha_{-i})$ is the set of player i's best mixed strategies when the list of the other players' mixed strategies is α_{-i}.

A profile α^* of mixed strategies is a mixed strategy Nash equilibrium if and only if every player's mixed strategy is a best response to the other players' mixed strategies:

- The mixed strategy profile α^* is a mixed strategy Nash equilibrium if and only if α^*_i is in $B_i(\alpha^*_{-i})$ for every player i.

Notes

Best Response Functions

In any two-player game in which each player has two actions like the Matching Pennies game, the best response function of each player consists either of a single pure strategy or of all mixed strategies. This has to do with the form of the payoff functions.

Consider the following strategic form game.

Notes

Best Response Functions

1. Players: $N = \{1, 2\}$
2. Actions: $A_1 = \{\text{T, B}\}$, $A_2 = \{\text{L, R}\}$.
3. vNM preferences captured by a Bernoulli payoff function u_i.

Player 1's mixed strategy α_1 assigns probability $\alpha_1(T)$ to her action T and probability $\alpha_1(B)$ to her action B.

For convenience, let $p = \alpha_1(T)$, so that $\alpha_1(B) = 1 - p$.

Similarly, denote the probability $\alpha_2(L)$ that player 2's mixed strategy assigns to L by q, so that $\alpha_2(R) = 1 - q$.

Notes

Best Response Functions

The probability distribution generated by the mixed strategy pair (α_1, α_2) over the four possible outcomes of the game is shown below:

Figure: Mixed Strategies

			Player 2	
			q	**1-q**
			L	R
Player 1	**p**	T	**pq**	**p(1-q)**
	1-p	B	**(1-p)q**	**(1-p)(1-q)**

The players' choices are assumed to be independent.

Notes

Best Response Functions

Figure: Mixed Strategies

			Player 2	
			q	1-q
			L	R
Player 1	p	T	pq	p(1-q)
	1-p	B	(1-p)q	(1-p)(1-q)

From this probability distribution, we can see that player 1's expected utility to the mixed strategy pair $(\alpha_1; \alpha_2)$ is:

$$\begin{aligned} E[u_1(\alpha)] &= pq \cdot u_1(T,L) + p(1-q) \cdot u_1(T,R) \\ &+ (1-p)q \cdot u_1(B,L) + (1-p)(1-q) \cdot u_1(B,R) \\ &= p[q \cdot u_1(T,L) + (1-q) \cdot u_1(T,R)] \\ &+ (1-p)[q \cdot u_1(B,L) + (1-q) \cdot u_1(B,R)] \end{aligned}$$

Notes

Best Response Functions

$$\begin{aligned} E[u_1(\alpha)] &= p[q \cdot u_1(T,L) + (1-q) \cdot u_1(T,R)] \\ &+ (1-p)[q \cdot u_1(B,L) + (1-q) \cdot u_1(B,R)] \end{aligned}$$

- The first term in square brackets is player 1's expected utility when she uses a pure strategy that assigns probability 1 to *T* and player 2 uses her mixed strategy α_2
- The second term in square brackets is player 1's expected utility when she uses a pure strategy that assigns probability 1 to *B* and player 2 uses her mixed strategy α_2.

Notes

Best Response Functions

If we denote these two expected utilities as $E_1(T, \alpha_2)$ and $E_1(B, \alpha_2)$, then we can write player 1's expected utility to the mixed strategy pair $(\alpha_1; \alpha_2)$ as

$$pE_1(T, \alpha_2) + (1-p)E_1(B, \alpha_2)$$

Player 1's expected payoff is a weighted average of her expected payoffs to T and B when player 2 uses the mixed strategy α_2, with weights equal to the probabilities assigned to T and B by α_1.

We can see that player 1's expected payoff, given player 2's mixed strategy, is a **linear** function of p.

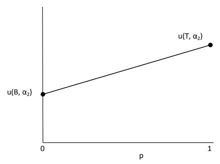
Notes

Best Response Functions

A significant implication of the linearity of player 1's expected payoff is that there are three possibilities for her best response to a given mixed strategy of player 2.

Possibility 1: Player 1's unique best response is the pure strategy T if $E_1(T, \alpha_2) > E_1(B, \alpha_2)$

Figure: Mixed Strategies

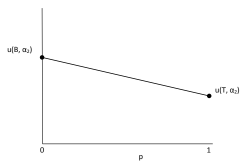


Notes

Best Response Functions

Possibility 2: Player 1's unique best response is the pure strategy B if $E_1(T, \alpha_2) < E_1(B, \alpha_2)$.

Figure: Mixed Strategies

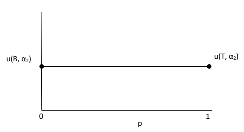


Notes

Best Response Functions

Possibility 3: All mixed strategies of player 1 yield the same expected payoff, and hence all best responses, if $E_1(T, \alpha_2) = E_1(B, \alpha_2)$.

Figure: Mixed Strategies



A mixed strategy $(p, 1-p)$ for which $0 \leq p \leq 1$ is never a unique best response; either it is not a best response or all mixed strategies are best responses.

Notes

Example: Battle of the Sexes

Consider the two-player game with vNM preferences in which the players' preferences over deterministic action profiles are the same as in the Battle of the Sexes and their preferences over lotteries are represented by the expected value of the payoff functions shown in the figure below.

What are the mixed strategy Nash equilibria of this game?

Figure: Battle of the Sexes

			Player 2	
			q	**1-q**
			Boxing	Ballet
Player 1	**p**	Boxing	3, 2	1, 1
	1-p	Ballet	0, 0	2,3

Notes

Example: Battle of the Sexes

We first construct player 1's best response function. To get this, we must calculate the expected utility of *Boxing*

$$\begin{aligned} E_1(Boxing, \alpha_2) &= 3 \cdot q + 1 \cdot (1-q) \\ &= 3q + 1 - q = 2q + 1 \end{aligned}$$

and the expected utility of *Ballet*

$$\begin{aligned} E_1(Ballet, \alpha_2) &= 0 \cdot q + 2 \cdot (1-q) \\ &= 2 - 2q \end{aligned}$$

$E_1(Boxing, \alpha_2) > E_1(Ballet, \alpha_2)$ if

$$\begin{aligned} 2q + 1 &> 2 - 2q \\ 4q &> 1 \\ q &> \frac{1}{4} \end{aligned}$$

Notes

Example: Battle of the Sexes

- If $q > \frac{1}{4}$ i.e. $E_1(Boxing, \alpha_2) > E_1(Ballet, \alpha_2)$, then player 1's unique best response is Boxing i.e. $p = 1$.
- If $q < \frac{1}{4}$ i.e. $E_1(Boxing, \alpha_2) < E_1(Ballet, \alpha_2)$, then player 1's unique best response is Ballet i.e. $p = 0$.
- If $q = \frac{1}{4}$ i.e. $E_1(Boxing, \alpha_2) = E_1(Ballet, \alpha_2)$, then all of player 1's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq p \leq 1$.

Thus, player 1's best response function is:

$$B_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{4} \\ p : 0 \leq p \leq 1 & \text{if } q = \frac{1}{4} \\ p = 1 & \text{if } q > \frac{1}{4} \end{cases}$$

Notes

Example: Battle of the Sexes

We can repeat this for player 2's best response function. The expected utility of *Boxing*

$$\begin{aligned} E_2(Boxing, \alpha_1) &= 2 \cdot p + 0 \cdot (1-p) \\ &= 2p \end{aligned}$$

and the expected utility of *Ballet*

$$\begin{aligned} E_2(Ballet, \alpha_1) &= 1 \cdot p + 3 \cdot (1-p) \\ &= p + 3 - 3p = 3 - 2p \end{aligned}$$

$E_2(Boxing, \alpha_1) > E_2(Ballet, \alpha_1)$ if

$$\begin{aligned} 2p &> 3 - 2p \\ 4p &> 3 \\ p &> \frac{3}{4} \end{aligned}$$

Notes

Example: Battle of the Sexes

- If $p > \frac{3}{4}$ i.e. $E_2(Boxing, \alpha_1) > E_2(Ballet, \alpha_1)$, then player 2's unique best response is Boxing i.e. $q = 1$.
- If $p < \frac{3}{4}$ i.e. $E_2(Boxing, \alpha_1) < E_2(Ballet, \alpha_1)$, then player 2's unique best response is Ballet i.e. $q = 0$.
- If $p = \frac{3}{4}$ i.e. $E_2(Boxing, \alpha_1) = E_2(Ballet, \alpha_1)$, then all of player 2's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq q \leq 1$.

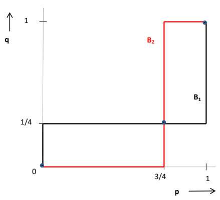
Thus, player 2's best response function is:

$$B_2(p) = \begin{cases} q = 0 & \text{if } p < \frac{3}{4} \\ q : 0 \leq q \leq 1 & \text{if } p = \frac{3}{4} \\ q = 1 & \text{if } p > \frac{3}{4} \end{cases}$$

Notes

Example: Battle of the Sexes

Figure: Battle of the Sexes



There are three mixed strategy Nash equilibria

1. $((0,1);(0,1))$ – degenerate mixed strategy where they each play pure strategies (*Ballet*; *Ballet*).
2. $\left(\left(\frac{3}{4},\frac{1}{4}\right);\left(\frac{1}{4},\frac{3}{4}\right)\right)$ – a proper mixed strategy.
3. $((1,0);(1,0))$ – degenerate mixed strategy where they each play pure strategies (*Boxing*; *Boxing*).

Notes

Example: Matching Pennies

Consider the two-player game with vNM preferences in which the players' preferences over deterministic action profiles are the same as in the Matching Pennies game and their preferences over lotteries are represented by the expected value of the payoff functions shown in the figure below.

What are the mixed strategy Nash equilibria of this game?

Figure: Matching Pennies

		Player 2: Heads	Player 2: Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

Notes

Example: Matching Pennies

We first construct player 1's best response function. To get this, we must calculate the expected utility of *Heads*

$$\begin{aligned} E_1(Heads, \alpha_2) &= 1 \cdot q + (1-q) \cdot (-1) \\ &= 2q - 1 \end{aligned}$$

and the expected utility of *Tails*

$$\begin{aligned} E_1(Tails, \alpha_2) &= (-1) \cdot q + 1 \cdot (1-q) \\ &= 1 - 2q \end{aligned}$$

$E_1(Heads, \alpha_2) > E_1(Tails, \alpha_2)$ if

$$\begin{aligned} 2q - 1 &> 1 - 2q \\ 4q &> 2 \\ q &> \frac{1}{2} \end{aligned}$$

Notes

Example: Matching Pennies

- If $q > \frac{1}{2}$ i.e. $E_1(Heads, \alpha_2) > E_1(Tails, \alpha_2)$, then player 1's unique best response is Heads i.e. $p = 1$.
- If $q < \frac{1}{2}$ i.e. $E_1(Heads, \alpha_2) < E_1(Tails, \alpha_2)$, then player 1's unique best response is Tails i.e. $p = 0$.
- If $q = \frac{1}{2}$ i.e. $E_1(Heads, \alpha_2) = E_1(Tails, \alpha_2)$, then all of player 1's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq p \leq 1$.

Thus, player 1's best response function is:

$$B_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p : 0 \leq p \leq 1 & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

Notes

Example: Matching Pennies

We can repeat this for player 2's best response function. The expected utility of *Heads*

$$\begin{aligned} E_2(Heads, \alpha_1) &= (-1) \cdot p + 1 \cdot (1-p) \\ &= 1 - 2p \end{aligned}$$

and the expected utility of *Tails*

$$\begin{aligned} E_2(Tails, \alpha_1) &= 1 \cdot p + (-1) \cdot (1-p) \\ &= 2p - 1 \end{aligned}$$

$E_2(Heads, \alpha_1) > E_2(Tails, \alpha_2)$ if

$$\begin{aligned} 1 - 2p &> 2p - 1 \\ 2 &> 4p \\ p &< \frac{1}{2} \end{aligned}$$

Notes

Example: Matching Pennies

- If $p > \frac{1}{2}$ i.e. $E_2(Heads, \alpha_2) < E_2(Tails, \alpha_2)$, then player 2's unique best response is Tails i.e. $q = 0$.
- If $p < \frac{1}{2}$ i.e. $E_2(Heads, \alpha_2) > E_2(Tails, \alpha_2)$, then player 2's unique best response is Heads i.e. $q = 1$.
- If $p = \frac{1}{2}$ i.e. $E_2(Heads, \alpha_2) = E_2(Tails, \alpha_2)$, then all of player 2's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq q \leq 1$.

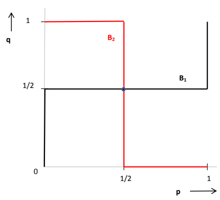
Thus, player 2's best response function is:

$$B_2(p) = \begin{cases} q = 0 & \text{if } p > \frac{1}{2} \\ q : 0 \leq q \leq 1 & \text{if } p = \frac{1}{2} \\ q = 1 & \text{if } p < \frac{1}{2} \end{cases}$$

Notes

Example: Matching Pennies

Figure: Battle of the Sexes



There is only one mixed strategy Nash equilibria

1. $((\frac{1}{2}, \frac{1}{2}); (\frac{1}{2}, \frac{1}{2}))$ – a proper mixed strategy.

We can see that Matching Pennies has no equilibrium if the players are not allowed to randomize.

Notes

Example: Hawk-Dove Game

Consider the two-player game with vNM preferences in which the players' preferences over deterministic action profiles are the same as in the Hawk-Dove game and their preferences over lotteries are represented by the expected value of the payoff functions shown in the figure below.

What are the mixed strategy Nash equilibria of this game?

Figure: Hawk-Dove Game

			Animal 2	
			q Aggressive	1-q Passive
Animal 1	p	Aggressive	0, 0	6, 1
	1-p	Passive	1, 6	3, 3

Notes

Example: Hawk-Dove Game

We first construct player 1's best response function. To get this, we must calculate the expected utility of *Aggressive*

$$\begin{aligned} E_1(Aggressive, \alpha_2) &= 0 \cdot q + (1-q) \cdot 6 \\ &= 6 - 6q \end{aligned}$$

and the expected utility of *Passive*

$$\begin{aligned} E_1(Passive, \alpha_2) &= 1 \cdot q + 3 \cdot (1-q) \\ &= 3 - 2q \end{aligned}$$

$E_1(Aggressive, \alpha_2) > E_1(Passive, \alpha_2)$ if

$$\begin{aligned} 6 - 6q &> 3 - 2q \\ 3 &> 4q \\ q &< \frac{3}{4} \end{aligned}$$

Notes

Example: Hawk-Dove Game

- If $q > \frac{3}{4}$ i.e. $E_1(Aggressive, \alpha_2) < E_1(Passive, \alpha_2)$, then player 1's unique best response is Passive i.e. $p = 0$.
- If $q < \frac{3}{4}$ i.e. $E_1(Aggressive, \alpha_2) > E_1(Passive, \alpha_2)$, then player 1's unique best response is Aggressive i.e. $p = 1$.
- If $q = \frac{3}{4}$ i.e. $E_1(Aggressive, \alpha_2) = E_1(Passive, \alpha_2)$, then all of player 1's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq p \leq 1$.

Thus, player 1's best response function is:

$$B_1(q) = \begin{cases} p = 0 & \text{if } q > \frac{3}{4} \\ p : 0 \leq p \leq 1 & \text{if } q = \frac{3}{4} \\ p = 1 & \text{if } q < \frac{3}{4} \end{cases}$$

Notes

Example: Hawk-Dove Game

We can repeat this for player 2's best response function. The expected utility of *Aggressive*

$$\begin{aligned} E_2(Aggressive, \alpha_1) &= 0 \cdot p + 6 \cdot (1-p) \\ &= 6 - 6p \end{aligned}$$

and the expected utility of *Passive*

$$\begin{aligned} E_2(Passive, \alpha_1) &= 1 \cdot p + 3 \cdot (1-p) \\ &= 3 - 2p \end{aligned}$$

$E_2(Aggressive, \alpha_1) > E_2(Passive, \alpha_2)$ if

$$\begin{aligned} 6 - 6p &> 3 - 2p \\ 3 &> 4p \\ p &< \frac{3}{4} \end{aligned}$$

Notes

Example: Hawk-Dove Game

- If $p > \frac{3}{4}$ i.e. $E_2(Aggressive, \alpha_2) < E_2(Passive, \alpha_2)$, then player 2's unique best response is Passive i.e. $q = 0$.
- If $p < \frac{3}{4}$ i.e. $E_2(Aggressive, \alpha_2) > E_2(Passive, \alpha_2)$, then player 2's unique best response is Aggressive i.e. $q = 1$.
- If $p = \frac{3}{4}$ i.e. $E_2(Aggressive, \alpha_2) = E_2(Passive, \alpha_2)$, then all of player 2's mixed strategies yield the same utility and so are all best responses i.e. $0 \le q \le 1$.

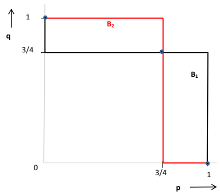
Thus, player 2's best response function is:

$$B_2(p) = \begin{cases} q = 0 & \text{if } p > \frac{3}{4} \\ q : 0 \le q \le 1 & \text{if } p = \frac{3}{4} \\ q = 1 & \text{if } p < \frac{3}{4} \end{cases}$$

Notes

Example: Hawk-Dove Game

Figure: Hawk-Dove Game



There are three mixed strategy Nash equilibria

1. $((0,1);(1,0))$ – degenerate mixed strategy where they each play pure strategies (*Passive*; *Aggressive*).
2. $\left(\left(\frac{3}{4},\frac{1}{4}\right);\left(\frac{3}{4},\frac{1}{4}\right)\right)$ – a proper mixed strategy.
3. $((1,0);(0,1))$ – degenerate mixed strategy where they each play pure strategies (*Aggressive*; *Passive*).

Notes

Example: Football Game

The setup of the Football Game is

- Players: $N = \{OC, DC\}$, where OC is offensive coordinator and DC is defensive coordinator.
- Actions: $OC = \{$pass, run$\}$, $DC = \{$pass defense, run defense$\}$.
  - The set of action profiles are $a = \{$(pass; pass defense) $=$ stop, (pass; run defense) $=$ touchdown, (run; pass defense) $=$ first down, (run; run defense) $=$ stop$\}$.
- Preferences
  - OC: (pass; run defense) $>$ (run; pass defense) $>$ (pass; pass defense) $=$ (run; run defense)
  - DC: (pass; pass defense) $=$ (run; run defense) $>$ (run; pass defense) $>$ (pass; run defense)

Notes

Example: Football Game

Figure: Football Game

			DC	
			q	1-q
			Pass Defense	Run Defense
OC	p	Pass	0, S	T, -T
	1-p	Run	F, -F	0, S

There are no Nash equilibria in pure strategies and so we need to look for mixed strategy NE.

1. Find all Nash equilibria.
2. What happens when the value of the touchdown increases?

Notes

Example: Football Game

We first construct the OC's best response function. To get this, we must calculate the expected utility of *Pass*

$$\begin{aligned} E_{OC}(Pass, \alpha_2) &= 0 \cdot q + (1-q) \cdot T \\ &= T - Tq \end{aligned}$$

and the expected utility of *Run*

$$\begin{aligned} E_{OC}(Passive, \alpha_2) &= F \cdot q + 0 \cdot (1-q) \\ &= qF \end{aligned}$$

$E_{OC}(Pass, \alpha_2) > E_{OC}(Run, \alpha_2)$ if

$$\begin{aligned} T - Tq &> qF \\ T &> qF + qT \\ T &> q(F+T) \\ \frac{T}{T+F} &> q \end{aligned}$$

Notes

Example: Football Game

- If $q > \frac{T}{T+F}$ i.e. $E_{OC}(Pass, \alpha_{DC}) < E_{OC}(Run, \alpha_{DC})$, then player 1's unique best response is Run i.e. $p = 0$.
- If $q < \frac{T}{T+F}$ i.e. $E_{OC}(Pass, \alpha_{DC}) > E_{OC}(Run, \alpha_{DC})$, then player 1's unique best response is Pass i.e. $p = 1$.
- If $q = \frac{T}{T+F}$ i.e. $E_{OC}(Pass, \alpha_{DC}) = E_{OC}(Run, \alpha_{DC})$, then all of player 1's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq p \leq 1$.

Thus, the OC's best response function is:

$$B_{OC}(q) = \begin{cases} p = 0 & \text{if } q > \frac{T}{T+F} \\ p : 0 \leq p \leq 1 & \text{if } q = \frac{T}{T+F} \\ p = 1 & \text{if } q < \frac{T}{T+F} \end{cases}$$

Notes

Example: Football Game

We can repeat this for the DC's best response function. The expected utility of *Pass Defense*

$$\begin{aligned} E_{DC}(Pass\ Defense, \alpha_1) &= S \cdot p + (-F) \cdot (1 - p) \\ &= pS + pF - F \end{aligned}$$

and the expected utility of *Run Defense*

$$\begin{aligned} E_{DC}(Run\ Defense, \alpha_1) &= (-T) \cdot p + S \cdot (1 - p) \\ &= S - pT - pS \end{aligned}$$

$E_{DC}(Pass\ Defense, \alpha_1) > E_2(Run\ Defense, \alpha_2)$ if

$$\begin{aligned} pS + pF - F &> S - pT - pS \\ p(S + F) + p(T + S) &> S + F \\ p(2S + F + T) &> S + F \\ p &> \frac{S + F}{2S + F + T} \end{aligned}$$

Notes

Example: Football Game

- If $p > \frac{S+F}{2S+F+T}$ i.e. $E_{DC}(Aggressive, \alpha_{OC}) < E_2(Passive, \alpha_{OC})$, then player 2's unique best response is Pass Defense i.e. $q = 1$.
- If $p < \frac{S+F}{2S+F+T}$ i.e. $E_{DC}(Aggressive, \alpha_{OC}) > E_2(Passive, \alpha_{OC})$, then player 2's unique best response is Run Defense i.e. $q = 0$.
- If $p = \frac{S+F}{2S+F+T}$ i.e. $E_{DC}(Aggressive, \alpha_{OC}) = E_2(Passive, \alpha_{OC})$, then all of player 2's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq q \leq 1$.

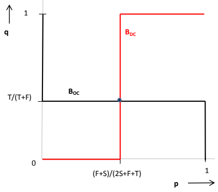
Thus, the DC's best response function is:

$$B_{DC}(p) = \begin{cases} q = 0 & \text{if } p < \frac{S+F}{2S+F+T} \\ q : 0 \leq q \leq 1 & \text{if } p = \frac{S+F}{2S+F+T} \\ q = 1 & \text{if } p > \frac{S+F}{2S+F+T} \end{cases}$$

Notes

Example: Football Game

Figure: Football Game



There is one mixed strategy Nash equilibrium

1. $\left(\left(\frac{S+F}{2S+F+T}, 1-\frac{S+F}{2S+F+T}\right); \left(\frac{T}{T+F}, 1-\frac{T}{T+F}\right)\right)$

Notes

Example: Football Game

The model/equilibrium analysis is your theory.

You get testable implications from the comparative statics.

Comparative statics indicate how the equilibrium changes as you change one of the model's parameters.

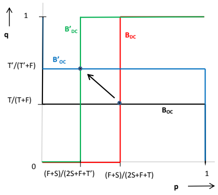
So, let's look at what happens when the value of a touchdown (T) increases.

What happens if $T \to T' : T' > T$?

Notes

Example: Football Game

Figure: Football Game



As the value of T increases, the probability of passing (p) decreases but the probability of pass defense (q) increases!

Empirically, as the end of the game approaches, the value of a touchdown increases. What we see is a lot of running plays by the offense and the defense pulls back because it doesn't want to allow touchdowns.

Notes

An Alternative to Best Response Functions

So far we have found the set of mixed strategy NE by constructing the players' best response functions. But there is another way.

You will have noticed in all the previous games that the proper mixed strategy NE occurs when the expected payoffs to each of a player's pure strategies are equal.

Thus, one way to find the proper mixed strategy NE is to set the expected payoffs of the player's strategies equal to each other and solve for the respective probabilities.

Notes

An Alternative to Best Response Functions

Definition: A mixed strategy profile a^* in a strategic game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium if and only if, for each player i

- the expected payoff, given a^*_{-i}, to every action to which a^*_i assigns positive probability is the same
- the expected payoff, given a^*_{-i}, to every action to which a^*_i assigns zero probability is at most the expected payoff to any action to which a^*_i assigns positive probability.

Each player's expected payoff in an equilibrium is her expected payoff to any of her actions that she uses with positive probability.

Notes

An Alternative to Best Response Functions

Consider the Battle of the Sexes game again.

Figure: Battle of the Sexes

			Player 2	
			q	**1-q**
			Boxing	Ballet
Player 1	**p**	Boxing	3, 2	1, 1
	1-p	Ballet	0, 0	2,3

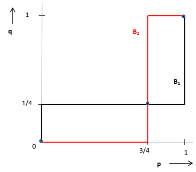
Notes

An Alternative to Best Response Functions

When constructing the best response functions for each player, we found the following:

- If $q = \frac{1}{4}$ i.e. $E_1(Boxing, \alpha_2) = E_1(Ballet, \alpha_2)$, then all of player 1's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq p \leq 1$.
- If $p = \frac{3}{4}$ i.e. $E_2(Boxing, \alpha_1) = E_2(Ballet, \alpha_1)$, then all of player 2's mixed strategies yield the same utility and so are all best responses i.e. $0 \leq q \leq 1$.

Figure: Battle of the Sexes



Notes

Best Response Functions and Indifference

The key then to finding the proper mixed strategy NE is simply to set the expected payoffs to each of a player's pure strategies equal to each other and solve for either p or q.

In effect, each player must choose either p or q to make the other player indifferent between her two actions.

Figure: Battle of the Sexes

			Player 2	
			q	**1-q**
			Boxing	Ballet
Player 1	**p**	Boxing	3, 2	1, 1
	1-p	Ballet	0, 0	2,3

Notes

Battle of the Sexes Again

The expected utility of *Boxing* for Player 1 is

$$E_1(Boxing, \alpha_2) = 3 \cdot q + 1 \cdot (1 - q)$$
$$= 3q + 1 - q = 2q + 1$$

and the expected utility of *Ballet*

$$E_1(Ballet, \alpha_2) = 0 \cdot q + 2 \cdot (1 - q)$$
$$= 2 - 2q$$

Notes

Battle of the Sexes Again

Player 2 needs to choose a q such that Player 1 is indifferent between $Boxing$ and $Ballet$.

$E_1(Boxing, \alpha_2) = E_1(Ballet, \alpha_2)$ if

$$\begin{aligned} 2q + 1 &= 2 - 2q \\ 4q &= 1 \\ q &= \frac{1}{4} \end{aligned}$$

So, in the proper mixed strategy NE, player 2 uses the mixed strategy $(q, 1-q) = (\frac{1}{4}, \frac{3}{4})$.

Notes

Battle of the Sexes Again

The expected utility of *Boxing* for Player 2 is

$$\begin{aligned} E_2(Boxing, \alpha_1) &= 2 \cdot p + 0 \cdot (1-p) \\ &= 2p \end{aligned}$$

and the expected utility of *Ballet*

$$\begin{aligned} E_2(Ballet, \alpha_1) &= 1 \cdot p + 3 \cdot (1-p) \\ &= p + 3 - 3p = 3 - 2p \end{aligned}$$

Notes

Battle of the Sexes Again

Player 1 needs to choose a p such that Player 2 is indifferent between $Boxing$ and $Ballet$.

$E_2(Boxing, \alpha_1) = E_2(Ballet, \alpha_1)$ if

$$\begin{aligned} 2p &= 3 - 2p \\ 4p &= 3 \\ p &= \frac{3}{4} \end{aligned}$$

So, in the proper mixed strategy NE, player 1 uses the mixed strategy $(p, 1-p) = (\frac{3}{4}, \frac{1}{4})$.

Notes

Battle of the Sexes Again

Player 1 needs to choose a p such that Player 2 is indifferent between $Boxing$ and $Ballet$.

$E_2(Boxing, \alpha_1) = E_2(Ballet, \alpha_1)$ if

$$\begin{aligned} 2p &= 3 - 2p \\ 4p &= 3 \\ p &= \frac{3}{4} \end{aligned}$$

So, in the proper mixed strategy NE, player 1 uses the mixed strategy $(p, 1-p) = (\frac{3}{4}, \frac{1}{4})$.

And so the proper mixed strategy NE of the Battle of the Sexes game is $((\frac{3}{4}, \frac{1}{4}); (\frac{1}{4}, \frac{3}{4}))$.

Notes

Matching Pennies Again

Figure: Matching Pennies

		Player 2	
		Heads	Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

The expected utility of *Heads* for Player 1 is

$$\begin{aligned} E_1(Heads, \alpha_2) &= 1 \cdot q + (1-q) \cdot (-1) \\ &= 2q - 1 \end{aligned}$$

and the expected utility of *Tails* is

$$\begin{aligned} E_1(Tails, \alpha_2) &= (-1) \cdot q + 1 \cdot (1-q) \\ &= 1 - 2q \end{aligned}$$

Notes

Matching Pennies Again

Player 2 needs to choose a q such that Player 1 is indifferent between $Heads$ and $Tails$.

$E_1(Heads, \alpha_2) = E_1(Tails, \alpha_2)$ if

$$\begin{aligned} 2q - 1 &= 1 - 2q \\ 4q &= 2 \\ q &= \frac{1}{2} \end{aligned}$$

So, in the proper mixed strategy NE, player 2 uses the mixed strategy $(q, 1-q) = (\frac{1}{2}, \frac{1}{2})$.

Notes

Matching Pennies Again

Figure: Matching Pennies

		Player 2 Heads	Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

The expected utility of *Heads* for Player 2 is

$$\begin{aligned} E_2(Heads, \alpha_1) &= (-1) \cdot p + 1 \cdot (1-p) \\ &= 1 - 2p \end{aligned}$$

and the expected utility of *Tails* is

$$\begin{aligned} E_2(Tails, \alpha_1) &= 1 \cdot p + (-1) \cdot (1-p) \\ &= 2p - 1 \end{aligned}$$

Notes

Matching Pennies Again

Player 1 needs to choose a p such that Player 2 is indifferent between $Heads$ and $Tails$.

$E_2(Heads, \alpha_1) = E_2(Tails, \alpha_2)$ if

$$\begin{aligned} 1 - 2p &= 2p - 1 \\ 2 &= 4p \\ p &= \frac{1}{2} \end{aligned}$$

So, in the proper mixed strategy NE, player 1 uses the mixed strategy $(p, 1-p) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Notes

Matching Pennies Again

Player 1 needs to choose a p such that Player 2 is indifferent between $Heads$ and $Tails$.

$E_2(Heads, \alpha_1) = E_2(Tails, \alpha_2)$ if

$$\begin{aligned} 1 - 2p &= 2p - 1 \\ 2 &= 4p \\ p &= \frac{1}{2} \end{aligned}$$

So, in the proper mixed strategy NE, player 1 uses the mixed strategy $(p, 1-p) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

And so the proper mixed strategy NE of the Matching Pennies game is $\left(\left(\frac{1}{2}, \frac{1}{2}\right); \left(\frac{1}{2}, \frac{1}{2}\right)\right)$.

Notes

Hawk-Dove Game Again

Figure: Hawk-Dove Game

			Animal 2	
			q Aggressive	1-q Passive
Animal 1	p	Aggressive	0, 0	6, 1
	1-p	Passive	1, 6	3, 3

The expected utility of *Aggressive* for Player 1 is

$$\begin{aligned} E_1(Aggressive, \alpha_2) &= 0 \cdot q + (1-q) \cdot 6 \\ &= 6 - 6q \end{aligned}$$

and the expected utility of *Passive* is

$$\begin{aligned} E_1(Passive, \alpha_2) &= 1 \cdot q + 3 \cdot (1-q) \\ &= 3 - 2q \end{aligned}$$

Notes

Hawk-Dove Game Again

Player 2 needs to choose a q such that Player 1 is indifferent between $Aggressive$ and $Passive$.

$E_1(Aggressive, \alpha_2) = E_1(Passive, \alpha_2)$ if

$$\begin{aligned} 6 - 6q &= 3 - 2q \\ 3 &= 4q \\ q &= \frac{3}{4} \end{aligned}$$

So, in the proper mixed strategy NE, player 2 uses the mixed strategy $(q, 1-q) = (\frac{3}{4}, \frac{1}{4})$.

Notes

Hawk-Dove Game Again

Figure: Hawk-Dove Game

			Animal 2	
			q Aggressive	1-q Passive
Animal 1	p	Aggressive	0, 0	6, 1
	1-p	Passive	1, 6	3, 3

The expected utility of *Aggressive* for Player 2 is

$$\begin{aligned} E_2(Aggressive, \alpha_1) &= 0 \cdot p + 6 \cdot (1-p) \\ &= 6 - 6p \end{aligned}$$

and the expected utility of *Passive* is

$$\begin{aligned} E_2(Passive, \alpha_1) &= 1 \cdot p + 3 \cdot (1-p) \\ &= 3 - 2p \end{aligned}$$

Notes

Hawk-Dove Game Again

Player 1 needs to choose a p such that Player 2 is indifferent between $Aggressive$ and $Passive$.

$E_2(Aggressive, \alpha_1) = E_2(Passive, \alpha_2)$ if

$$\begin{aligned} 6 - 6p &= 3 - 2p \\ 3 &= 4p \\ p &= \frac{3}{4} \end{aligned}$$

So, in the proper mixed strategy NE, player 1 uses the mixed strategy $(p, 1-p) = \left(\frac{3}{4}, \frac{1}{4}\right)$.

Notes

Hawk-Dove Game Again

Player 1 needs to choose a p such that Player 2 is indifferent between $Aggressive$ and $Passive$.

$E_2(Aggressive, \alpha_1) = E_2(Passive, \alpha_2)$ if

$$\begin{aligned} 6 - 6p &= 3 - 2p \\ 3 &= 4p \\ p &= \frac{3}{4} \end{aligned}$$

So, in the proper mixed strategy NE, player 1 uses the mixed strategy $(p, 1-p) = \left(\frac{3}{4}, \frac{1}{4}\right)$.

And so the proper mixed strategy NE of the Matching Pennies game is $\left(\left(\frac{3}{4}, \frac{1}{4}\right); \left(\frac{3}{4}, \frac{1}{4}\right)\right)$.

Notes

Soccer Penalty Game

We can easily extend this strategy to games in which players have any number of finite actions.

Consider a soccer penalty where the shooter must overcome the opponent team's goalkeeper. The shooter may target the left side (L), the center (C), or the right side (R) of the goal.

If the goalkeeper moves to the target of the penalty shooter, she (or her team) gets a payoff of 1 and the shooter (or her team) gets zero.

If the goalkeeper does not move to the shooter's target, she gets zero and the shooter gets a payoff of 1.

Find the unique Nash equilibrium of the game. What are the expected equilibrium payoffs of the penalty shooter, i.e., what proportion of her shots will yield a goal?

Notes

Soccer Penalty Game

Figure: Soccer Penalty Game

		Goalkeeper		
		L	C	R
	L	0, $\underline{1}$	$\underline{1}$, 0	$\underline{1}$, 0
Shooter	C	$\underline{1}$, 0	0, $\underline{1}$	$\underline{1}$, 0
	R	$\underline{1}$, 0	$\underline{1}$, 0	0, $\underline{1}$

There are no NE in pure strategies. But what about a proper mixed strategy NE?

Notes

Soccer Penalty Game

Figure: Soccer Penalty Game

			Goalkeeper		
			q_L	q_C	q_R
			L	C	R
	p_L	L	0, 1	1, 0	1, 0
Shooter	p_C	C	1, 0	0, 1	1, 0
	p_R	R	1, 0	1, 0	0, 1

We know that $p_L + p_C + p_R = 1$ and $q_L + q_C + q_R = 1$.

Notes

Soccer Penalty Game

To derive the equilibrium, we must derive the expected payoffs for each player for each of the three actions and set them equal.

Figure: Soccer Penalty Game

			Goalkeeper		
			q_L	q_C	q_R
			L	C	R
	p_L	L	0, 1	1, 0	1, 0
Shooter	p_C	C	1, 0	0, 1	1, 0
	p_R	R	1, 0	1, 0	0, 1

- $E_S[L, q] = q_C + q_R$
- $E_S[C, q] = q_L + q_R$
- $E_S[R, q] = q_L + q_C$

Using $E_S[L, q] = E_S[C, q] = E_S[R, q]$ and $q_L + q_C + q_R = 1$, it is easy to see that $q_L = q_C = q_R = \frac{1}{3}$.

Notes

Soccer Penalty Game

Figure: Soccer Penalty Game

			Goalkeeper		
			q_L	q_C	q_R
			L	C	R
Shooter	p_L	L	0, 1	1, 0	1, 0
	p_C	C	1, 0	0, 1	1, 0
	p_R	R	1, 0	1, 0	0, 1

Similarly for the goalkeeper, it is easy to see that

- $E_G[L, p] = p_L$
- $E_G[C, p] = p_C$
- $E_G[R, p] = p_R$

Using $E_G[L, p] = E_G[C, p] = E_G[R, p]$ and $p_L + p_C + p_R = 1$, it is easy to see that $p_L = p_C = p_R = \frac{1}{3}$.

Notes

Soccer Penalty Game

Thus, the unique NE of the Soccer Penalty Game is $\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right); \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$.

Given these probabilities, the shooter will make a goal on average in 2 out 3 penalties for any of her three actions (i.e., $E_S[L, q] = q_C + q_R = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ and similarly $E_S[C, q] = \frac{2}{3}$ and $E_S[R, q] = \frac{2}{3}$. Hence, in equilibrium the shooter will make a goal in 2 out of 3 penalties (i.e., the proportion is $\frac{2}{3}$).

Notes

Notes